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THIN CURVILINEAR BEAMS OF MINIMUM WEIGHT*

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The problem of minimizing the weight of thin curvilinear beams whose state of stress is described by the Saint-Venant theory, is considered. The elastic state of the beam is analyzed and the influence of the shape of the cross-section on the state of stress studied. It is assumed that every point of the beam must satisfy some condition of the strength of material. Solutions of concrete problems are given for beams of varying configuration. Problems of minimum weight of beams and bars described by the equations of the elementary theory of bending, were discussed in /1—3/.

1. Formulation of the problem.

Let us consider a curvilinear spatially situated beam of length l. We attach to the beam axis a curvilinear x_1, x_2, x_3 -coordinate system (Fig. 1). Let the unit vector $\mathbf{r}_3$ be directed along the tangent to the beam axis, and unit vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ lying in the plane orthogonal to the $\mathbf{r}_3$ plane of the cross-section of the beam, be directed along the principal axes of inertia of the beam. We shall assume the coordinate axes x_1 and x_2 to be straight lines, so that the Lamé coefficients $X_1 = X_2 = 1$. We denote the arc length along the coordinate axis x_3 by r_3 $(dr_3 = X_3 dx_3)$.

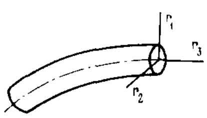

Fig.1

Let us consider the cross-section of the beam. We shall assume that the shape of the cross-section depends on the coordinate x_3 in such a manner that the linear dimensions of the cross-section are proportional to the similarity coefficient $\theta(x_3)$, with the center of similarity coinciding with the center of inertia of the beam. In this case the similarity coefficient must satisfy the inequality

$$\theta \geqslant 0 \tag{1.1}$$

The cross-section obtained for $\theta = 1$ will be called the initial cross-section. Clearly, not all forms of the initial cross-section have similar cross-sections for any $\theta > 0$. However, all convex cross-sections and a majority of the cross-sections which have practical applications, have similar cross-sections at any θ.

Let us denote the displacements, angles of rotation, moments, forces, external distributed loads and moment vectors by $\mathbf{u}$, $\boldsymbol{\varphi}$, $\mathbf{M}$, $\mathbf{P}$, $\mathbf{p}$, $\mathbf{m}$, respectively. Then the equations of equilibrium of a curvilinear beam will have the form (a prime denotes a derivative with respect to r_3) /4/

$$\mathbf{P}' = -\mathbf{p}, \quad \mathbf{M}' = -\mathbf{m} - \mathbf{r}_3 \times \mathbf{P}, \quad \mathbf{u}' = -\mathbf{r}_3 \times \boldsymbol{\varphi} + \mathbf{A}\cdot\mathbf{P}, \quad \boldsymbol{\varphi}' = \mathbf{C}\cdot\mathbf{M} \tag{1.2}$$

Here $\mathbf{A}$ and $\mathbf{C}$ are second rank tensors depending on the form of the cross-section and the elastic constants of the beam material. In a coordinate system associated with the principle axes of inertia of the system, the above tensors assume the diagonal form

$$A_{kk} = \frac{1}{Gsi_k}, \quad C_{kk} = \frac{1}{Ej_k} \quad (k = 1, 2), \quad A_{33} = \frac{1}{Es}, \quad C_{33} = \frac{1}{Gc} \tag{1.3}$$

where E is Young's modulus, G is shear modulus, s is the area, c is the torsional rigidity j_1 and j_2 are the principal moments of inertia and i_1, i_2 are the shear coefficients of the beam's cross-section. The relations (1.3) show that (the zero subscript denotes the values referring to the initial cross-section)

$$A_{kk}{}^0 = \theta^2 A_{kk}, \quad C_{kk}{}^0 = \theta^4 C_{kk} \quad (k = 1, 2, 3) \tag{1.4}$$

Using the configuration of the axis as a criterion, we can separate all beams into two groups. The first group will contain all open beams. The boundaries of such beams are $r_3 = 0$ and $r_3 = l$, and we set the following conditions at these boundaries:

$$-a_{1k}P_k(0) + b_{1k}u_k(0) - g_{1k} = 0, \quad a_{2k}P_k(l) + b_{2k}u_k(l) - g_{2k} = 0 \tag{1.5}$$

$$-c_{1k}M_k(0) + d_{1k}\varphi_k(0) - f_{1k} = 0, \quad c_{2k}M_k(l) + d_{2k}\varphi_k(l) - f_{2k} = 0 \quad (k = 1, 2, 3)$$

*Prikl.Matem.Mekhan.,44,No.4,720-726,1980 where a_{nk}, b_{nk}, c_{nk}, d_{nk}, g_{nk}, f_{nk} $(n=1, 2; k=1, 2, 3)$ are given coefficients and $a_{nk}^2+b_{nk}^2\neq 0$, $c_{nk}^2+d_{nk}^2\neq 0$. The coefficients f_{nk} and g_{nk} may represent either concentrated forces, or moments acting at the beam ends, or given displacements and angles of rotation. The boundary conditions (1.5) are sufficiently general and embrace a large class of real structures. The second group contains closed beams. In this case there will be no boundary conditions and the integration constants can be found from the conditions of periodicity of $\mathbf{u}$, $\boldsymbol{\varphi}$, $\mathbf{M}$ and $\mathbf{P}$.

We obtain the stresses within the beam using the Saint-Venant theory /5/

$$\sigma_{31}=\frac{P_1G}{Ei_2}\left(\frac{\partial\chi_1}{\partial x_1}-x_1^2\right)+\frac{P_2G}{Ei_1}\left(\frac{\partial\chi_2}{\partial x_1}-2\nu x_1x_2\right)+\frac{M_3}{c}\frac{\partial\chi_3}{\partial x_2},\quad \sigma_{32}=\frac{P_1G}{Ei_2}\left(\frac{\partial\chi_1}{\partial x_2}-2\nu x_1x_2\right)+\frac{P_2G}{Ei_1}\left(\frac{\partial\chi_2}{\partial x_2}-x_2^2\right)-\frac{M_3}{c}\frac{\partial\chi_3}{\partial x_1} \quad (1.6)$$

$$\sigma_{33}=\frac{M_1x_2}{i_1}-\frac{M_2x_1}{i_2}+\frac{P_3}{s},\quad \sigma_{11}=\sigma_{12}=\sigma_{22}=0$$

Here ν is the Poisson's ratio and $\chi_k=\chi_k(x_1, x_2)$ $(k=1, 2, 3)$ are solutions of the boundary value problems

$$\Delta\chi_1=0,\quad \partial\chi_1/\partial r_1'|_\Gamma=\alpha_1x_1^2+2\nu\alpha_2x_1x_2,\quad \Delta\chi_2=0,\quad \partial\chi_2/\partial r_1'|_\Gamma=2\nu\alpha_1x_1x_2+\alpha_2x_2^2 \quad (1.7)$$

$$\Delta\chi_3=-2,\quad \partial\chi_3/\partial r_2'|_\Gamma=0$$

where we introduce the following notation: Δ is the Laplace operator, Γ is the contour encircling the cross section S, x_1', x_2', x_3' denote a system of curvilinear coordinates attached to the contour Γ in such a manner that the unit vector $\mathbf{r}_3'$ is parallel to the vector $\mathbf{r}_3$, $\mathbf{r}_2'$ is directed along the tangent and $\mathbf{r}_1'$ along the normal to the contour Γ, α_1 and α_2 are direction cosines of the normal r_1', and r_1', r_2' are the arc lengths along the coordinate lines x_1' and x_2'.

The stresses (1.6) can be expressed in terms of the stresses in the initial cross-section Indeed, let us change the coordinates x_1, x_2 and functions χ_1, χ_2, χ_3 as follows:

$$x_k=\theta x_k^0,\quad \chi_k=\theta^3\chi_k^0(x_1^0, x_2^0)\ (k=1, 2),\quad \chi_3=\theta^2\chi_3^0(x_1^0, x_2^0) \quad (1.8)$$

where the zero superscript denotes, as before, the quantities associated with the initial cross-section. Substituting (1.8) into the differential equations and boundary conditions (1.7), we obtain the boundary value problems for the quantities accompanied by the superscript zero.

Let us introduce into our discussion the following functions defined on the set $x_1^0, x_2^0\in s^0$:

$$\chi_{nk}=\frac{G}{Ej_k^0}\left(\frac{\partial\chi_n^0}{\partial x_n^0}-x_n^{02}\right),\quad \chi_{nk}=\frac{G}{Ej_n^0}\left(\frac{\partial\chi_k^0}{\partial x_n^0}-2\nu x_1^0x_2^0\right),\quad (n, k=1, 2;\ n\neq k)$$

$$\chi_{13}=\frac{1}{c^0}\frac{\partial\chi_3^0}{\partial x_2^0},\quad \chi_{23}=-\frac{1}{c^0}\frac{\partial\chi_3^0}{\partial x_1^0},\quad \chi_{31}=\frac{x_2^0}{j_2^0},\quad \chi_{32}=-\frac{x_1^0}{j_1^0},\quad \chi_{33}=\frac{1}{s^0}$$

Substituting the right-hand sides of (1.8) into the expressions for the stresses (1.6), we obtain the following convenient formulas

$$\sigma_{3k}=\chi_{k1}P_1/\theta^2+\chi_{k2}P_2/\theta^2+\chi_{k3}M_3/\theta^3,\quad \sigma_{33}=\chi_{31}M_1/\theta^3+\chi_{32}M_2/\theta^3+\chi_{33}P_3/\theta^2,\quad (k=1, 2) \quad (1.9)$$

As the condition of material strength, we consider the following inequality:

$$-I_2(\operatorname{Dev}\boldsymbol{\sigma})\leqslant r_0^2 \quad (1.10)$$

where $I_2(\operatorname{Dev}\boldsymbol{\sigma})$ is the second invariant of the deviator of the stress tensor $\boldsymbol{\sigma}$, and τ_0 is the torsional yield point of the material. Substituting the stresses (1.9) into the strength of material condition (1.10), we obtain

$$\omega=\max_{x_1^0, x_2^0\in s^0}[(\chi_{31}M_1/\theta^3+\chi_{32}M_2/\theta^3+\chi_{33}P_3/\theta^2)^2/3+(\chi_{11}P_1/\theta^2+\chi_{12}P_2/\theta^2+\chi_{13}M_3/\theta^3)^2+ \quad (1.11)$$

$$(\chi_{21}P_1/\theta^2+\chi_{22}P_2/\theta^2+\chi_{23}M_3/\theta^3)^2]-\tau_0^2\leqslant 0$$

In addition to the constraints (1.11) which are called, in the optimal control, the zero order constraints /6/, a necessity often arises for the constraints

$$0\leqslant\lambda_1\leqslant\theta\leqslant\lambda_2 \quad (1.12)$$

when justified by design arguments.

Let us pose the following optimization problem. To find, amongst the piecewise continuous controls $\theta(r_3)$ and peicewise smooth functions $\mathbf{u}(r_3)$ $\boldsymbol{\varphi}(r_3)$, $\mathbf{M}(r_3)$, $\mathbf{P}(r_3)$, satisfying the equations (1.2), boundary conditions (1.5) and constraints (1.11), (1.12), such controls and functions which would minimize the weight of the beam

$$J=\int_0^l\rho g s^0\theta^2\,dr_3 \quad (1.13)$$

Here g is acceleration due to gravity and ρ is the density of the beam material independent, as a rule, of the coordinate r_3.

2. Necessary conditions of optimality. The problem of minimum weight of the beam refers to the problem of optimal control with fixed ends, with the zero order constraints (1.11) imposed on the phase coordinates, and constraints (1.12) imposed on the control. Following /7/, we pass from the inequality-type constraints to the equality-type constraints

$$\omega + \theta_1^2 = 0, \quad -\theta + \lambda_1 + \theta_2^2 = 0, \quad \theta - \lambda_2 + \theta_3^2 = 0 \tag{2.1}$$

where θ_1, θ_2 and θ_3 are supplementary control functions.

Let us construct a Hamiltonian function H /8/. To do this we add to the integrand expression in (1.13) the right-hand sides of the equations (1.2) scalar multiplied, one after the other, by the undefined Lagrange multipliers $-\mathbf{v}$, $-\boldsymbol{\psi}$, $\mathbf{Q}$, $\mathbf{N}$, and the left-hand sides of the equations (2.1) multiplied by the Lagrange multipliers μ_1, μ_2, μ_3

$$\begin{gathered} H = \rho g s^0 \theta^2 + \mathbf{v}\cdot\mathbf{p} + \boldsymbol{\psi}\cdot(\mathbf{m} + \mathbf{r}_3 \times \mathbf{P}) + \mathbf{Q}\cdot(-\mathbf{r}_3 \times \boldsymbol{\varphi} + \mathbf{A}\cdot\mathbf{P}) + \\ \mathbf{N}\cdot(\mathbf{C}\cdot\mathbf{M}) + \mu_1(\omega + \theta_1^2) + \mu_2(-\theta + \lambda_1 + \theta_2^2) + \mu_3(\theta + \lambda_2 + \theta_3^2) \end{gathered} \tag{2.2}$$

Using the results of /8/ we obtain, for the control problem in question, the Euler equations (as before, the prime denotes a derivative with respect to r_3)

$$\mathbf{Q}' = \mathbf{0}, \quad \mathbf{N}' + \mathbf{r}_3 \times \mathbf{Q} = \mathbf{0}, \quad \mathbf{v}' + \mathbf{r}_3 \times \boldsymbol{\psi} = \mathbf{A}\cdot\mathbf{Q} + \mu_1 \frac{\partial\omega}{\partial\mathbf{P}} \tag{2.3}$$

$$\boldsymbol{\psi}' = \mathbf{C}\cdot\mathbf{N} + \mu_1 \frac{\partial\omega}{\partial\mathbf{M}}, \quad 2\rho g s^0 \theta + \mathbf{Q}\cdot\frac{\partial\mathbf{A}}{\partial\theta}\cdot\mathbf{P} + \mathbf{N}\cdot\frac{\partial\mathbf{C}}{\partial\theta}\cdot\mathbf{M} + \mu_1\frac{\partial\omega}{\partial\theta} - \mu_2 + \mu_3 = 0, \quad 2\mu_k\theta_k = 0 \quad (k = 1, 2, 3)$$

Analysis of the above equations shows that the first four Euler equations formally coincide with the equations (1.2), provided that we perform the analogy between $\mathbf{u}$ and $\mathbf{v}$, $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$, $\mathbf{M}$ and $\mathbf{N}$, $\mathbf{P}$ and $\mathbf{Q}$. Analizing the last three Euler equations we find, that they are equivalent to the equations

$$\mu_1\omega = 0, \quad \mu_2(-\theta + \lambda_1) = 0, \quad \mu_3(\theta - \lambda_2) = 0$$

The Weierstrass—Erdman conditions for the problem of minimization of weight reduce to

$$[\mathbf{v}] = \mathbf{0}, \quad [\boldsymbol{\psi}] = \mathbf{0}, \quad [\mathbf{N}] = \mathbf{0}, \quad [\mathbf{Q}] = \mathbf{0}, \quad [H] = 0, \quad [F] = F(r_3 + 0) - F(r_3 - 0) \tag{2.4}$$

Taking into account the first four conditions of (2.4) we obtain, from $[H] = 0$,

$$\rho g s^0 [\theta]^2 + \mathbf{v}\cdot[\mathbf{p}] + \boldsymbol{\psi}\cdot[\mathbf{m}] + \mathbf{Q}\cdot\mathbf{A}^0\cdot\mathbf{P}\,[1/\theta^2] + \mathbf{N}\cdot\mathbf{C}^0\cdot\mathbf{M}\,[1/\theta^4] = 0 \tag{2.5}$$

Let us now construct a function obtained by summing the left-hand parts of the boundary conditions (1.5) multiplied by the undefined Lagrange multipliers $\xi_{1k}, \xi_{2k}, \eta_{1k}, \eta_{2k}$ respectively. Using the results of /8/ we obtain, at the points $r_3 = 0$ and $r_3 = l$, twenty four conditions

$$Q_k(0) = -\xi_{1k}b_{1k}, \quad v_k(0) = -\xi_{1k}a_{1k}, \quad N_k(0) = -\eta_{1k}d_{1k}, \quad \psi_k(0) = -\eta_{1k}c_{1k} \tag{2.6}$$

$$Q_k(l) = \xi_{2k}b_{2k}, \quad v_k(l) = -\xi_{2k}a_{2k}, \quad N_k(l) = \eta_{2k}d_{2k}, \quad \psi_k(l) = \eta_{2k}c_{2k} \quad (k = 1, 2, 3)$$

twelve of which are used for determining the Lagrange multipliers $\xi_{1k}, \xi_{2k}, \eta_{1k}, \eta_{2k}$ and the remaining twelve represent the boundary conditions for the Euler equations (2.3). Eliminating from (2.6) the Lagrange multipliers, we obtain the boundary conditions in a more convenient form

$$\begin{gathered} -a_{1k}Q_k(0) + b_{1k}v_k(0) = 0, \quad a_{2k}Q_k(l) + b_{2k}v_k(l) = 0 \\ -c_{1k}N_k(0) + d_{1k}\psi_k(0) = 0, \quad c_{2k}N_k(l) + d_{2k}\psi_k(l) = 0 \\ (k = 1, 2, 3) \end{gathered} \tag{2.7}$$

Using the formulation of the present paper, we obtain the necessary Clebsch and Weierstrass conditions of weak and strong minimum in the form of the inequalities (2.8) and (2.9), respectively

$$\mu_1 \geqslant 0, \quad \mu_2 \geqslant 0, \quad \mu_3 \geqslant 0 \tag{2.8}$$

$$H(\mathbf{M}, \mathbf{P}, \mathbf{N}, \mathbf{Q}, \theta) \geqslant H(\mathbf{M}, \mathbf{P}, \mathbf{N}, \mathbf{Q}, \Theta) \tag{2.9}$$

where $\theta(r_3)$ is the optimal and $\Theta(r_3)$ any admissible control.

3. Optimal beam systems. Let us take, for simplicity, a circle of radius $a = 1$ m as the initial cross-section. Solving for this cross-section the boundary value problems (1.7) we obtain

$$\begin{gathered} \chi_{11} = \frac{(3 + 2\nu)(1 - x_1^{02}) - (1 - 2\nu)x_2^{02}}{2\pi(1 + \nu)}, \quad \chi_{12} = \chi_{21} = -\frac{(1 + 2\nu)x_1^0 x_2^0}{\pi(1 + \nu)} \\ \chi_{22} = \frac{(3 + 2\nu)(1 - x_2^{02}) - (1 - 2\nu)x_1^{02}}{2\pi(1 + \nu)}, \quad \chi_{13} = -\frac{4x_2^0}{\pi}, \quad \chi_{23} = \frac{4x_1^0}{\pi} \\ \chi_{31} = 4x_2^0/\pi, \ \chi_{32} = -4x_1^0/\pi, \ \chi_{33} = 1/\pi \end{gathered} \tag{3.1}$$

Minimum weight cantilever. The cantilever represents a rectilinear beam, one end of which is fixed and the other free. For a rectilinear beam the system of equations (1.2) splits into four systems of equations (bending in two planes, tension and torsion). The boundary conditions (1.5) for the cantilever simplify to

$$\mathbf{u}(0)=\boldsymbol{\varphi}(0)=0,\ P_k(l)=g_k,\ M_k(l)=f_k\ (k=1,2,3) \tag{3.2}$$

Solving the equations (1.2) with boundary conditions (3.2), we find

$$M_1=f_1-g_2(l-r_3)+\int_{r_3}^{l}m_1(\zeta)\,d\zeta-\int_{r_3}^{l}d\zeta\int_{\zeta}^{l}p_2(\zeta)\,d\zeta \tag{3.3}$$

$$M_2=f_2+g_1(l-r_3)+\int_{r_3}^{l}m_2(\zeta)\,d\zeta+\int_{r_3}^{l}d\zeta\int_{\zeta}^{l}p_1(\zeta)\,d\zeta$$

$$M_3=f_3+\int_{r_3}^{l}m_3(\zeta)\,d\zeta,\quad P_k=g_k+\int_{r_3}^{l}p_k(\zeta)\,d\zeta\quad (k=1,2,3)$$

The control $\theta(r_3)$ is determined by the equation

$$\omega=0 \tag{3.4}$$

Let us take $p_k=m_k=0,\ f_k=0\ (k=1,2,3),\ g_1=g_3=0,\ g_2=2.14\cdot10^4\,\mathrm{N}$, $l=1$ m, $\nu=0.25$, $\tau_0=1.47\cdot10^8\,\mathrm{N/m^2}$ $\rho=7.8\cdot10^3\,\mathrm{kg/m^3}$.

The optimal control is shown in Fig.2 by a solid line. The weight of the optimal beam and the constant cross-section beam satisfying the strength of material condition (1.11) are, respectively, $J_0\approx220$ N, $J_\text{п}\approx384$ N (weight payoff $\approx43\%$)

Minimum weight of a beam with a helical axis. We take a beam with a circular cross-section, the axis of which represents a helix. One end of the beam is fixed, and the other end is acted upon by a force g (Fig.3). Let us put $l=1$ m. as the length of a single turn of the helix, $h=0.1$ m. as one pitch of th helix, $g=9.8\cdot10^3$ N, $\tau_0=1.47\cdot10^8\,\mathrm{N/m^2}$, $\nu=0.25$, $\rho=7.8\cdot10^3\,\mathrm{kg/m^3}$ Solving the equations (1.2) we obtain

$$P_1=0,\quad P_2=-\frac{gd}{l},\quad P_3=\frac{gb}{l},\quad M_1=\frac{gl}{2\pi}\sin\frac{2\pi(l-r_3)}{l}$$

$$M_2=-\frac{gdb}{2\pi l}\left[1-\cos\frac{2\pi(l-r_3)}{l}\right],\quad M_3=-\frac{gd^2}{2\pi l}\left[1-\cos\frac{2\pi(l-r_3)}{l}\right],\quad d=\sqrt{l^2-h^2},\ b=d/(2\pi)$$

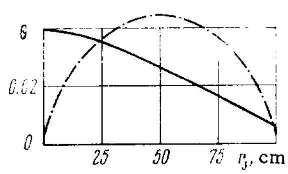

Fig.2

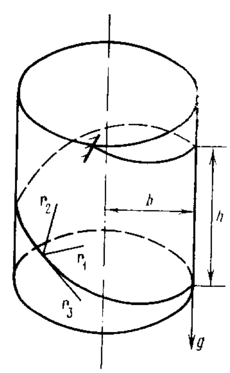

Fig.3

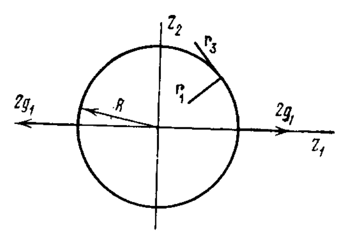

Fig.4

The optimal control $\theta(r_3)$ can be found from equation (3.4) and is represented in Fig.2 by a dashed line. The weight of the optimal beam and the constant cross-section beam satisfying the strength of material condition are, respectively, $J_0\approx289$ N, $J_\text{п}\approx473$ N (weight payoff is $\approx40\%$).

Minimum weight of a beam with a circular axis. We take a beam of circular cross-section, with a circular axis (Fig.4), acted upon by concentrated tensile forces $2g_1$. Let us put $b=0.1$ m, $g_1=9.8\cdot10^3$ N, $\tau_0=1.47\cdot10^8\,\mathrm{N/m^2}$, $E=2.06\cdot10^{11}\,\mathrm{N/m^2}$, $\nu=0.31$, $\rho=7.8\cdot10^3\,\mathrm{kg/m^3}$.

Since the problem of stretching a circular ring has symmetry axes $z_1=0$ and $z_2=0$, we shall consider a quarter of the ring $r_3\in[0,\pi b/2]$ with boundary conditions

$$u_3(0)=\varphi_2(0)=u_3(l)=\varphi_2(l)=P_1(l)=0,\ P_1(0)=g_1$$

An exact solution of the problem in question cannot be obtained. We have, therefore, solved the problem in which the restriction (1.11) was regarded as a multiplier, imposing a penalty on the functional (1.13). The optimal beam and the constant cross-section beam satisfying the strength of material condition (1.11) are shown in Fig.5 by the solid lines, and the distribution of the moment M_2 by a dashed line. The weights of the optimal and the constant cross-section beams are, respectively, $J_0 \approx 5.98$ N, $J_{\text{п}} \approx 11.0$ N (weight payoff $\approx 46\%$).

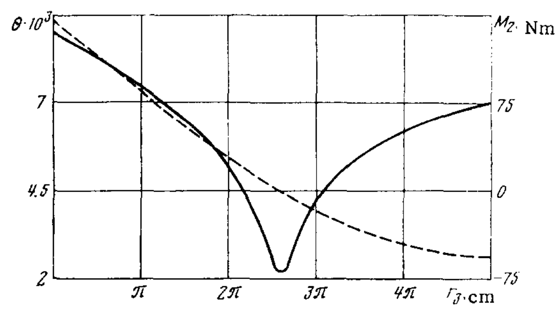

Fig.5

Analysing the optimal beams we find that all these beams have equal strength irrespective of the form of the beam axis. We note that a beam is called equistrong if every cross-section contains a point at which the restriction (1.1) becomes an equality. The points need not be situated at some specified part of the cross-section, they may change their position within the cross-section depending on the arc length r_3.

The process of solving the problem could be somewhat simplified if it could be shown that the formulation of the problem with the inequality-type constraints (1.11) is equivalent to the problem with the corresponding, equality-type constraints.

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